Differential equation

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Definition 1. A *differential equation* (DE) is an equation which involves derivatives. An *ordinary differential equation* (ODE) is a differential equation in which there is exactly one independent variable. A *partial differential equation* (PDE) is one where there are at least two independent variables. The derivatives of an ODE are ordinary-, whereas those of a PDE are partial derivatives.

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Definition 2. Consider a differential equation. The *order* of it is the order of the highest derivative appearing in it. Its *degree* is the degree of the highest ordered derivative therein. A *primitive* is a relation between the variables that involves n essential arbitrary constants, which gives rise to a differential equation of order n. The n constants are called *essential* if they cannot be replaced by a smaller number of constants.

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Example 1. The differential equation $y''' + 3\left(y''\right)^2 + 2y' = \sin x$ is an ordinary differential equation of order 3 and degree one. The differential equation $\left(y''\right)^2 + \left(y'\right)^3 + y = 2x$ is an ODE which has an order 2 and degree 2.

Problem 1. The problem of finding solutions of differential equations is essentially that of finding the primitive which gave rise to the equation.

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Example 2. The differential equation $y''' = 0$ has a primitive $y = Ax^2 + Bx + C$, $y''' - 6y'' + 11y' - 6y = 0$ has $y = C_1e^{3x} + C_2e^{2x} + C_3e^x$, $y^2\left(y''\right)^2 + y^2 = r^2$ has $(x-C)^2 + y^2 = r^2$.

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Definition 3. *Existence theorems* give conditions by which one could determine whether a differential equation is solvable. A *particular solution* of a differential equation is one obtained from the primitive by assigning definite values to the parameters, that is to say, the arbitrary constants. A *singular solution* is a solution which cannot be obtained from the primitive by any manipulation of the arbitrary constants. The primitive of a differential equation is usually called the *general solution* of the equation.

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Definition 4. A differential equation is said to be *variable separable* if an integrating factor can be readily found. Such equation has the form

$$f_2(x) \cdot g_2(y)\,\mathrm{d}\,x + f_2(x) \cdot g_1(y)\,\mathrm{d}\,y = 0$$

Through the use of the integrating factor

$$\frac{1}{f_2(x) \cdot g_2(y)}$$

the primitive of this is then

$$\int \frac{f_1(x)}{f_2(x)}\,\mathrm{d}\,x + \int \frac{g_1(y)}{g_2(y)}\,\mathrm{d}\,y = C$$

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Definition 5. A differential equation of the first order and first degree may be written in the form

$$M(x,y)\,\mathrm{d}\,x + N(x,y)\,\mathrm{d}\,y = 0$$

If this such equation admits a solution $f(x,y,C) = 0$ where C is an arbitrary constant, then there exist infinitely many integrating factors $xi(x,y)$ such that $\xi(x,y)\left[M(x,y)\,\mathrm{d}\,x + N(x,y)\,\mathrm{d}\,y\right] = 0$ is exact, and there exist transformations of the variables which render the latter separated. But since no general rules exist for doing this, the use in practice is still somewhat limited.

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Definition 6. A function $f(x,y)$ is said to be *homogeneous* of degree n if

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

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Note 1. The equation

$$(a_1 x + b_1 y + c_1)\,\mathrm{d}\,x + (a_2 x + b_2 y + c_2)\,\mathrm{d}\,y = 0$$

where $a_1 b_2 - a_2 b_1 = 0$, is reduced through the transformation

$$a_1 x + b_1 y = t \quad \text{and} \quad \mathrm{d}\,y = \frac{\mathrm{d}\,t - a_1\,\mathrm{d}\,x}{b_1}$$

to the form

$$P(x,t)\,\mathrm{d}\,x + Q(x,t)\,\mathrm{d}\,t = 0$$

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Note 2. The equation

$$(a_1 x + b_1 y + c_1)\,\mathrm{d}\,x + (a_2 x + b_2 y + c_2)\,\mathrm{d}\,y = 0$$

where $a_1 b_2 - a_2 b_1 \neq 0$, is reduced through the transformation

$$x = x' + h \quad \text{and} \quad y = y' + k$$

in which $x = h$ and $y = k$ are the solutions of the equations $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ into the homogeneous form

$$(a_1 x' + b_1 y')\,\mathrm{d}\,x' + (a_2 x' + b_2 y')\,\mathrm{d}\,y' = 0$$

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Note 3. The equation of the form

$$y \cdot f(xy)\,\mathrm{d}\,x + x \cdot g(xy)\,\mathrm{d}\,y = 0$$

through the transformation

$$xy = z, \qquad y = \frac{z}{x}, \qquad \mathrm{d}\,y = \frac{x\,\mathrm{d}\,z - z\,\mathrm{d}\,x}{x^2}$$

into the form

$$P(x,y)\,\mathrm{d}\,x + Q(x,z)\,\mathrm{d}\,z = 0$$

which is variable separable.

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Bibliography

Frank Ayres, Jr. *Theory and problems of Differential Equations.* Schaum's Outline Series, 1981(1952)